Exercises

Eigenvectors and Eigenvalues

Exercise 1. Let $A = \begin{pmatrix} 2 & 4 & 3 \\ 4 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$. Check if the vectors $a_1 = (0, -3, 4)^\top$, $a_2 = (5, 4, 3)^\top$, and $a_3 = (0, 3, 4)^\top$ are eigenvectors of the matrix A. Find all eigenvalues of A.

Exercise 2. Compute the eigenvalues and eigenvectors of the matrices

$$A = \begin{pmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} \text{ and } A^{-1}$$

Exercise 3. Depending on a, find the eigenvalues of the following matrices.

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 0 \end{pmatrix}$$

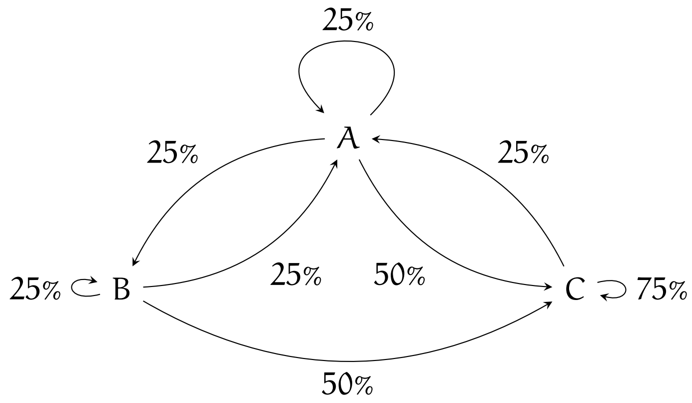
Exercise 4. Find the diagonalization of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and compute the matrix power A^5.

Exercise 5. For an arbitrary matrix $A \in \mathbb{R}^{n \times n}$ prove that A is invertible if and only if all eigenvalues of A are non-zero, i.e. $\lambda_i \neq 0$ for all eigenvalues λ_i.
Hint: Use that λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

Exercise 6. A rental car company has three bases in three different cities A, B, C. The customers can pick up and return cars at any station. For example, a customer could pick up his car at A and return it in C.
We assume that there is only one car type available and that the all customers rent their cars for one fixed time period (say one day). The following graph shows the proportions of cars that remain at a station or are transferred to other station over a day:



(For example: 20% of the cars at station B are returned at station A, 30% at station B and 50% at station C.)

1. Find an appropriate matrix $P \in \mathbb{R}^{3\times 3}$ that describes this transition process, i.e. a matrix P such that $P \cdot v_0$ gives the distribution of cars after one day, if v_0 is the initial distribution.

2. Suppose at the first day the distribution of all cars is $v_0 = \begin{pmatrix} 0.5 \\ 0.25 \\ 0.25 \end{pmatrix}$ (i.e. 50% of all cars are in A, 25% are in B and 25% are in C).

   (a) Compute the distribution of the cars after 1 day.

   (b) Compute the distribution of the cars after 2 days.

   (c) Compute the (approximate) distribution of the cars after 100 days.
   Hint: Use the diagonalization of P.

3. How should the cars be distributed initially such that the distribution does not change over time? (Find a distribution vector $v \in \mathbb{R}^3$ such that $Pv = v$.)